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Greater Boston Math Olympiad, 4th Grade, Solutions

1. (10 points) Solve:

$$\begin{array}{r} ABC \\ +\ BCA \\ \hline 719 \end{array}$$

(here ABC and BCA are 3-digit numbers with digits A,B,C, and different letters stand for different digits).

Answer: A = 2, B = 4, C = 7

We have A + B = 6 or 7 (left column) and C + A = 9 (right column), so C is at least 2.
Also, from the second column, B + C = 1 or 11. But since C is at least 2, we get B + C = 11
and hence A + B = 6 (not 7). Thus A + B + C = [(A + B) + (B + C) + (C + A)] / 2 = (6 + 11 + 9) / 2 = 13.
Subtracting, we get A = (A + B + C) – (B + C) = 2, and similarly B = 4 and C = 7.

2. Money in Wonderland comes in \$5 and \$7 bills.

(**a**) (6 points) What is the smallest amount of money you need to have in order to buy a slice of pizza which costs \$1 and get back your change in full? (The pizza man has plenty of \$5 and \$7 bills.) For example, having \$7 won't do, since the pizza man can only give you \$5 back.

Answer: \$15.

This part can be solved by trial and error, but there is also a more systematic solution (see the solution of (b)).

(**b**) (10 points) Vending machines in Wonderland accept only exact payments (do not give back change). List all positive integer numbers which CANNOT be used as prices in such vending machines. (That is, find the sums of money that cannot be paid by exact change.)

Answer: 1, 2, 3, 4, 6, 8, 9, 11, 13, 16, 18, 23

First of all, any amount of at least \$28 can be paid by exact change.

Indeed, let N > 27. Then one of the five numbers N, N – 7, N – 14, N – 21, N – 28 is divisible by 5. Indeed, they have the same remainders under division by 5 as N, N – 2, N – 4, N – 1, N – 3, which are 5 consecutive numbers and hence one of them must be divisible by 5. Thus we can pay \$N by paying N, N – 7, N – 14, N – 21, or N – 28 (whichever is divisible by 5) using \$5 bills, and paying the rest by \$7 bills.

So any amount that cannot be paid by exact change is less than \$28. Thus it remains to list all numbers of the form $5a + 7b$ between 0 and 27, where a,b are nonnegative integers; the missing numbers are the answer to (b). Clearly, it is enough to consider $a < 6$, $b < 4$ (as we are looking for numbers < 28). By a direct computation, we get the following list of numbers $5a + 7b$ less than 28 (in the increasing order):

0, 5, 7, 10, 12, 14, 15, 17, 19, 20, 21, 22, 24, 25, 26, 27 (*)

So the missing numbers (answer to (b)) are 1, 2, 3, 4, 6, 8, 9, 11, 13, 16, 18, 23

Now, to solve (a), it is enough to find the first two consecutive numbers in the list (*). These are 14 and 15. Thus the smallest amount to buy a \$1 pizza slice is \$15 (3 \$5-bills); the change will be two \$7 bills.

3. (8 points) John wrote down 2004 numbers: 1, 2, … , 2004. How many digits did he write?

Answer: 6909

If all these numbers had 4 digits there would be $4 \times 2004 = 8016$ digits. But some numbers have fewer digits. Namely, all numbers below 1000 miss 1 digit, all numbers below 100 miss 2 digits, and all numbers below 10 miss 3 digits. There are 9 numbers below 10, 99 numbers below 100, and 999 numbers below 1000. So the answer is $8016 – 9 – 99 – 999 = 6909$.

4. (8 points) How many zeros are there at the end of the number $100! = 1 \times 2 \times \ldots \times 100$?

Answer: 24

The number of zeros at the end of any number N is equal to minimum(m,k), where m is the number of times N is divisible by 2 and k is the number of times N is divisible by 5 (since a zero at the end is obtained if a 2 is multiplied by a 5). If N = 100! then the number k of times N is divisible by 5 is 24: there are 20 numbers between 1 and 100 which are divisible by 5, and 4 of them (25,50,75,100) are divisible by 5 twice. The number m of times N is divisible by 2 is at least 50, since there are 50 even numbers among 1, ... , 100. Thus, minimum(m,k) = 24, so N has 24 zeros at the end.

5. (8 points) Put 5 points on the plane so that each 3 of them are vertices of an isosceles triangle (i.e., a triangle with two equal sides), and no three points lie on the same line.

Answer: The 5 points can be put at vertices of a regular pentagon, or 4 of them at vertices of the regular pentagon and the fifth in the center of the pentagon.

6. (10 points) How many times in a half-day (= 12 hours) the hour and the minute hand of a clock form the right angle with each other?

Answer: 22

In 12 hours, the hour hand makes one full circle, while the minute hand makes 12 full circles.
If an observer "sits" on the hour hand, he'll see that the hour hand does not move (with respect to him), while the minute hand makes $12 - 1 = 11$ full circles (and the clock itself makes one circle in the backwards direction). While making each of these 11 circles, the minute hand makes right angle with the hour hand twice. So the answer is $2 \times 11 = 22$.