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Greater Boston Math Olympiad, 6th Grade, Solutions

1. (6 points) Solve: BOOK + BOOK + BOOK + BOOK + BOOK + BOOK = TEST
(BOOK and TEST are 4-digit numbers, and different letters stand for different digits).

Answer: BOOK = 1443, TEST = 8658

We have $6 \times$ BOOK = TEST. This means that B = 1 (if B > 1 then the product would have 5 digits).
Also, T has to be even (being the last digit of $6 \times$ K), and T is at least 6. At the same time, K has to be odd (otherwise K = T, since for an even number X, the numbers X and $6 \times$ X end with the same digit).
Finally, K is not equal to 1 (as B = 1). Thus, T cannot be 6, so T = 8 and K = 3. This immediately implies that O = 4, which gives the answer.

2. (6 points) The number A2...2B has 2004 digits (all digits standing between A and B are 2).
This number is divisible by 72. Find the digits A and B.

Answer: A = 6, B = 4

Since this number is divisible by $72 = 8 \times 9$, it is divisible by 8 and 9. A number ending with 200 is always divisible by 8, so the only digit B for which a number ...22B is divisible by 8 is B = 4. Now A should be chosen so that the number is divisible by 9. This happens if and only if the sum of its digits is divisible by 9. The sum of digits is $A + 2 \times 2002 + 4 = A + 4008$. The number 4008 has sum of digits 12, so it has remainder 3 when divided by 9. Thus A = 6 is the only digit making the number divisible by 9.

3. A number N is divisible by 18 and has exactly 10 divisors (including 1 and N).

(**a**) (7 points) Find such N.

(**b**) (8 points) Is it unique?

Answer: N=162 (and it is unique).

Let N be a positive integer. Let p be a prime and $k(N,p)$ be the number of times N is divisible by p.
Then any divisor of N has the form:

product over primes p dividing N of p in the power $m(p)$,where $m(p)$ is any number between 0 and $k(N,p)$.

So for each prime p, there are $k(N,p) + 1$ possibilities for $m(p)$.
Since the numbers $m(p)$ are chosen independently for all primes p, the number of divisors of N is

D(N) = product of numbers $k(N,p) + 1$ over all primes p dividing N. (*)

In our situation, $D(N) = 10 = 2 \times 5$, so there are at most two factors in the product (*), as all factors are at least 2. But there has to be at least two factors, since 2 and 3 divide N.
So there are exactly two factors. Since 9 divides N, we have:

$D(N)=(k(N,2) + 1) (k(N,3) + 1) = 10$ and $k(N,3) > 1$.

Hence $k(N,2) = 1$, $k(N,3) = 4$ and $N = 2 \times 3 \times 3 \times 3 \times 3 = 162$. So N is unique.

4. Two people play a game. They put 3 piles of matches on the table:
the first one contains 2 matches, the second one 3 matches, and the third one 4 matches.
Then they take turns making moves. In a move, a player may take any nonzero number of matches FROM ONE PILE. The player who takes the last match from the table loses the game.

(**a**) (5 points) The player who makes the first move can win the game.
What is the winning first move?

(**b**) (6 points) How can he win? (Describe his strategy.)

The answer can be found by analyzing who wins the game for smaller initial pile sizes (starting from very small and gradually increasing it). The sequence of pile sizes to consider which most quickly leads to the answer is

1,1,1; 0,2,2; 1,2,3; 2,3,4.

The first move should be taking 3 matches from the pile of 4, reducing pile sizes to 1,2,3. Then the second player will lose. Whatever move he makes, the first player can do one of three things on his second move (check this!):

1) Win immediately (by taking all matches but 1)

2) reduce the piles sizes to 1,1,1

3) reduce the pile sizes to 0,2,2

The positions 1,1,1 and 0,2,2 are clearly winning for the first player.
So in situations 2) and 3) he wins on the third move.

5. (**a**) (4 points) How many times in a day (= 24 hours) the hour and the minute hand of a clock form the right angle with each other?

Answer: 44

In 24 hours, the hour hand makes two full circles, while the minute hand makes 24 full circles.
If an observer "sits" on the hour hand, he'll see that the hour hand does not move (with respect to him), while the minute hand makes $24 - 2 = 22$ full circles (and the clock itself makes two circles in the backwards direction). While making each of these 22 circles, the minute hand makes right angle with the hour hand twice. So the answer is $2 \times 22 = 44$.

(**b**) (8 points) How many times in a day the seconds hand of a clock falls on the line bisecting the angle between the hour and the minute hands?

Answer: 2856.

In 24 hours, the hour hand makes two full circles, while the minute hand makes 24 full circles.
This means that the line bisecting the angle between them makes $(2 + 24) / 2 = 13$ full circles, which is $13 \times 2=26$ half-circles.

The seconds hand makes 1 circle a minute, so 60 an hour and $60 \times 24 = 1440$ in 24 hours. This amounts to $1440 \times 2 = 2880$ half-circles.

If an observer "sits" on the bisecting line, he'll see that the bisecting line does not move (with respect to him), while the seconds hand makes $2880 - 26 = 2856$ half-circles. While making each of these half-circles, the seconds hand aligns once with the bisecting line. So the answer is 2856.

6. (**a**) (3 points) Put 5 points on the plane so that each 3 of them are vertices of an isosceles triangle (i.e., a triangle with two equal sides), and no three points lie on the same line.

(**b**) (7 points) Do the same with 6 points.

Answers: The points are vertices of a regular pentagon and its center.