Greater Boston Math Olympiad, 5th Grade, Solutions

1. (6 points) Solve: INK + INK + INK + INK + INK + INK = PEN
(INK and PEN are 3-digit numbers, and different letters stand for different digits).

Answer: INK = 105, PEN = 630

We have $6 \times$ INK = PEN. This means that I = 1 (if I > 1 then the product would have 4 digits). For the same reason N is at most 6. Also, N has to be even (being the last digit of $6 \times$ K). At the same time, K has to be odd (otherwise K = N, since for an even number X, the numbers X and $6 \times$ X end with the same digit). Finally, K is not equal to 1 (as I = 1). Thus we have three possibilities to consider: N = 0, K = 5;
N = 2, K = 7; N = 4, K = 9.They give $6 \times 105 = 630$, $6 \times 127 = 762$, $6 \times 149 = 894$. Only the first one satisfies the condition that different letters stand for different digits.

2. (7 points) The number A2...2B has 2004 digits (all digits standing between A and B are 2).
This number is divisible by 72. Find the digits A and B.

Answer: A = 6, B = 4

Since this number is divisible by $72 = 8 \times 9$, it is divisible by 8 and 9. A number ending with 200 is always divisible by 8, so the only digit B for which a number ...22B is divisible by 8 is B = 4. Now A should be chosen so that the number is divisible by 9. This happens if and only if the sum of its digits is divisible by 9. The sum of digits is A + 2×2002 + 4 = A + 4008. The number 4008 has sum of digits 12, so it has remainder 3 when divided by 9. Thus A = 6 is the only digit making the number divisible by 9.

3. Money in Wonderland comes in \$5 and \$7 bills.

(**a**) (4 points) What is the smallest amount of money you need to have in order to buy a slice of pizza which costs \$1 and get back your change in full? (The pizza man has plenty of \$5 and \$7 bills.) For example, having \$7 won't do, since the pizza man can only give you \$5 back.

Answer: \$15.

This part can be solved by trial and error, but there is also a more systematic solution
(see the solution of (b)).

(**b**) (8 points) Vending machines in Wonderland accept only exact payments (do not give back change).

List all positive integer numbers which CANNOT be used as prices in such vending machines.
(That is, find the sums of money that cannot be paid by exact change.)

Answer: 1, 2, 3, 4, 6, 8, 9, 11, 13, 16, 18, 23
First of all, any amount of at least $28 can be paid by exact change.

Indeed, let $N > 27$. Then one of the five numbers $N, N-7, N-14, N-21, N-28$ is divisible by 5. Indeed, they have the same remainders under division by 5 as $N, N-2, N-4, N-1, N-3$, which are 5 consecutive numbers and hence one of them must be divisible by 5. Thus we can pay \$N by paying $N, N-7, N-14, N-21$, or $N-28$ (whichever is divisible by 5) using $5 bills,and paying the rest by $7 bills.

So any amount that cannot be paid by exact change is less than $28. Thus it remains to list all numbers of the form $5a + 7b$ between 0 and 27, where a,b are nonnegative integers; the missing numbers are the answer to (b). Clearly, it is enough to consider $a < 6$, $b < 4$ (as we are looking for numbers < 28). By a direct computation, we get the following list of numbers $5a + 7b$ less than 28 (in the increasing order):

0, 5, 7, 10, 12, 14, 15, 17, 19, 20, 21, 22, 24, 25, 26, 27 (*)

So the missing numbers (answer to (b)) are 1, 2, 3, 4, 6, 8, 9, 11, 13, 16, 18, 23

Now, to solve (a), it is enough to find the first two consecutive numbers in the list (*). These are 14 and 15. Thus the smallest amount to buy a $1 pizza slice is $15 (3 $5-bills); the change will be two $7 bills.

4. Two people play a game. They put 3 piles of matches on the table:
the first one contains 1 match, the second one 3 matches, and the third one 4 matches.
Then they take turns making moves. In a move, a player may take any nonzero number of matches FROM ONE PILE. The player who takes the last match from the table loses the game.

(**a**) (5 points) The player who makes the first move can win the game. What is the winning first move?

(**b**) (6 points) How can he win? (Describe his strategy.)

The answer can be found by analyzing who wins the game for smaller initial pile sizes (starting from very small and gradually increasing it). The sequence of pile sizes to consider which most quickly leads to the answer is

1,1,1; 0,2,2; 1,2,3; 1,3,4.

The first move should be taking 2 matches from the pile of 4, reducing pile sizes to 1,2,3. Then the second player will lose. Whatever move he makes, the first player can do one of three things on his second move (check this!):

1) Win immediately (by taking all matches but 1)

2) Reduce the piles sizes to 1,1,1

3) Reduce the pile sizes to 0,2,2

The positions 1,1,1 and 0,2,2 are clearly winning for the first player.

So in situations 2) and 3) he wins on the third move.

5. (**a**) (4 points) How many times in a half-day (= 12 hours) the hour and the minute hand of a clock form the right angle with each other?

Answer: 22

In 12 hours, the hour hand makes one full circle, while the minute hand makes 12 full circles.
If an observer "sits" on the hour hand, he'll see that the hour hand does not move (with respect to him), while the minute hand makes $12 - 1 = 11$ full circles (and the clock itself makes one circle in the backwards direction). While making each of these 11 circles, the minute hand makes right angle with the hour hand twice. So the answer is $2 \times 11 = 22$.

(**b**) (8 points) How many times in a half-day the seconds hand of a clock falls on the line bisecting the angle between the hour and the minute hands?

Answer: 1427

In 12 hours, the hour hand makes one full circle, while the minute hand makes 12 full circles.
This means that the line bisecting the angle between them makes
$(1 + 12) / 2 = 13 / 2$ full circles, which is 13 half-circles.

The seconds hand makes 1 circle a minute, so 60 an hour and $60 \times 12 = 720$ in 12 hours. This amounts to $720 \times 2 = 1440$ half-circles.

If an observer "sits" on the bisecting line, he'll see that the bisecting line does not move (with respect to him), while the seconds hand makes $1440 - 13 = 1427$ half-circles. While making each of these half-circles, the seconds hand aligns once with the bisecting line. So the answer is 1427.

6. (**a**) (4 points) Put 5 points on the plane so that each 3 of them are vertices of an isosceles triangle (i.e., a triangle with two equal sides), and no three points lie on the same line.

(**b**) (8 points) Do the same with 6 points.

Answers: The points are vertices of a regular pentagon and its center.